

# Conformal Properties of an Evaporating Black Hole Model

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## Abstract

We use a new, conformally-invariant method of analysis to test incomplete null geodesics approaching the singularity in a model of an evaporating black hole for the possibility of extensions of the conformal metric. In general, a local conformal extension is possible from the future but not from the past.

## 1 Introduction

In [7], Penrose developed his argument that, in the correct theory of space-time, initial singularities and final singularities would be different in character, with the Weyl tensor finite or even zero at initial singularities but no such constraint on final singularities. As a test case, he discussed the space-time singularities associated with Hawking's picture of an evaporating black hole (see also [8]). In this picture, there is a final singularity formed in the collapse which led to the black hole. At the endpoint of the evaporation of this black hole, this singularity vanishes but not before, at the last instant, being visible in the exterior and therefore being briefly an initial singularity. Penrose's argument then suggests that the singularity as seen from the past should be different in character, with respect to conformal properties, from the singularity as seen from the future.

To test this suggestion, we need a convincing model of the metric for an evaporating black hole and we shall use the model of Hiscock [3], [4]. Essentially, this consists of matching Vaidya metrics to the Schwarzschild metric of the black hole and to flat space, with an ingoing flux of negative energy density which reduces the Schwarzschild mass to zero, and an outgoing flux of positive energy density which radiates the mass to infinity, leaving flat space to the future of an outgoing null hypersurface. Details will be given below, but note that the model is spherically-symmetric.

Once we have the model, we test its conformal properties using a method developed in [5]. Null geodesics are conformally-invariant as point sets, in the sense of being unchanged under conformal rescalings of the metric, and as a point set a null geodesic is also a null conformal geodesic (see e.g. [11] for the general definition of conformal geodesic). By regarding a null metric geodesic, say  $\gamma$ , as a null conformal geodesic, one can give a conformally-invariant definition of propagation of frames along  $\gamma$  and can then give conformally-invariant definitions of boundedness of conformal curvature and its derivatives along  $\gamma$ , namely by requiring components of the (conformally-invariant) conformal curvature in these conformally-invariant frames to be bounded. In general, it is more natural to work with the calculus of tractors and tractor curvature (see [11]). For our purposes in this article this technology isn't needed, but we recall that the *tractor curvature*, which is the curvature of the tractor connection, has as nonzero components the Weyl tensor  $C_{abc}{}^d$  and the Cotton tensor, which is the derivative  $\nabla_{[a}P_{b]}{}^d$  of the Schouten tensor (defined below). Now suppose that  $\gamma$  is incomplete as a metric null geodesic but that there is a conformal rescaling of the metric which allows an extension of the space-time so that  $\gamma$  can be extended as a null geodesic of the rescaled metric. Then necessarily the conformal curvature (and the tractor curvature) will be bounded in the rescaled metric, so that the conformally-invariant conditions of boundedness will hold. (It is a recent result of one of us [6] that these necessary conditions are sufficient: given bounded derivatives up to order  $k+1$  of the tractor curvature, a  $C^k$  local extension of the conformal structure will exist.)

This method was used in [5] in a variety of particular cases: on radial null and space-like geodesics in the Schwarzschild metric, to find, as expected, that no conformal extension is possible through the singularities at  $r = 0$  or through space-like infinity  $i_0$ ; on radial null geodesics of the *conical spherical* metric:

$$ds^2 = dt^2 - dr^2 - a^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad a \neq 1,$$

(so-called because it has a singularity at the origin due to a deficit in the solid angle, analogous to the deficit in the plane angle at the familiar conical singularity) to show that no conformal extension is possible through the singularity at  $r = 0$ ; and along the matter flow-lines of the Lemaitre-Tolman-Bondi dust cosmologies to find conditions for conformal extendibility through the singularity.

In this article, we use the method to analyse the singularity which is formed in the collapse and then evaporates with the black hole. This singularity has, in the Penrose diagram (see figure 1), an interior and a final point before it vanishes. Our findings may be summarised as follows:

- along radial in or outgoing null geodesics meeting the singularity at an interior point, the physical affine parameter distance to the singularity is finite, and the Weyl spinor can never be made finite by rescaling; no conformal extension is possible (note that this includes regions of the Schwarzschild singularity, so in particular, as one would expect, no conformal extension is possible through there);
- along radial ingoing null geodesics from the past to the final point of the singularity, the physical affine parameter distance is finite; the Weyl spinor vanishes by continuity; the tractor curvature vanishes if one imposes Hiscock's condition ([3], [4]) of finite rates of particle creation; by imposing more conditions on the mass function

of the Vaidya interior metric, one may make more derivatives of the tractor curvature zero (and if all derivatives up to order  $k + 1$  are bounded then there is a local  $C^k$ -extension of the conformal structure, [6]); if one imposes the decay rate on the mass function which follows naively from the Hawking mass-loss formula, then the tractor curvature is singular (and no local extension of the conformal structure is possible);

- along radial outgoing null geodesics from the past to the final point of the singularity, the situation is similar to that with the conical spherical metric: there are choices of conformal factor which lead to a bounded Weyl spinor, but then the singularity is infinitely far off in the rescaled metric; all other choices lead to an unbounded Weyl tensor; either way no conformal extension is possible (it is also worth noting that the affine parameter distance to the singularity in the physical, unrescaled metric is infinite in this case);
- along past-directed, radial ingoing null geodesics from the future to the final point of the singularity the physical affine parameter distance is finite, and of course the tractor curvature and its derivatives are zero; local conformal extension is possible.

In summary, the conclusions accord with Penrose's hypothesis: a conformal extension is possible from the future but not from the past, with the exception that local extensions of low differentiability may be possible along ingoing null geodesics to the final point, given extra conditions on the rate at which the mass-function vanishes. A global extension from the past, through the whole singularity, is never possible.

The plan of the article is as follows. We begin in the next section by giving more details of the evaporating black hole (EBH) models, introducing a Newman-Penrose null tetrad and calculating what we need from the spin-coefficient formalism (for which see e.g. [9] or [10]). In section 3, we give the details of the method of testing for the possibility of a conformal extension, in terms of null conformal geodesics and the conformally-invariant propagation of spinors along them. In section 4, we collect the results of applying this test along radial in and outgoing null geodesics.

## 2 The model

As noted above, Hiscock's evaporating black hole (EBH) models [3], [4] are constructed by matching Vaidya metrics to the Schwarzschild metric of the black hole and to flat space. Start with a Schwarzschild metric, formed in collapse, and given to the future of a constant  $v$  surface  $\mathcal{S}$ , say  $v = v_S$ , in the usual coordinates (so  $v = t + r + 2m \log(r - 2m)$  where  $m$  is the mass of the initial Schwarzschild metric). The details of the collapse are assumed to lie to the past of  $\mathcal{S}$ . Pick a sphere  $S = (u = u_1, v = v_1)$  on  $\mathcal{S}' = \{v = v_1 > v_S\}$  and a spherically-symmetric time-like surface  $\Sigma$  with past boundary at  $S$ . In the space-time region to the future of  $\mathcal{S}'$  bounded by  $\Sigma$ , take the metric to be the ingoing Vaidya metric (1) below, with mass  $N(v)$ , where  $N(v_1) = m$  and  $N$  decreases to zero at  $v_0$  (so  $N(v_0) = 0$ ) as negative mass flows into the hole. In the space-time region to the future of  $\mathcal{S}'' = \{u = u_1\}$  bounded by  $\Sigma$ , take the metric to be the outgoing Vaidya metric (2) below, with mass  $M(u)$ , where  $M(u_1) = m$  and  $M$  decreases to zero at  $u_0$  (so  $M(u_0) = 0$ ) as positive mass flows out to future-null-infinity  $\mathcal{I}^+$ .  $\Sigma$  has future boundary at the 2-sphere

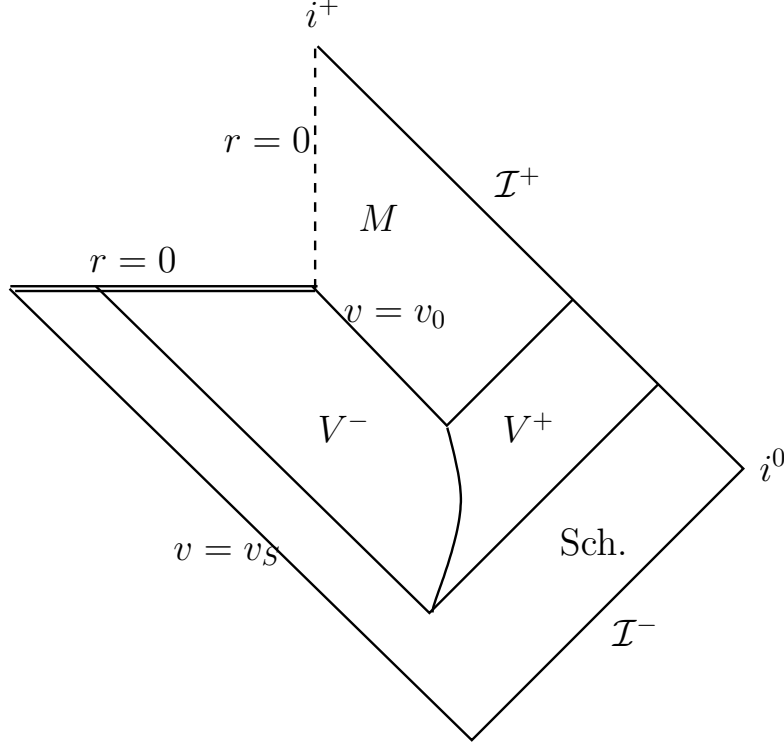


Figure 1: Penrose diagram of the model EBH space-time;  $M$ ,  $V^+$ ,  $V^-$  and Sch. are regions of Minkowski space, outgoing Vaidya, ingoing Vaidya and Schwarzschild respectively;  $V^+$  meets  $V^-$  along the time-like hypersurface  $\Sigma$ , other matchings are along null hypersurfaces ( $V^-$  to Sch. along  $v = v_1$ ,  $V^+$  to Sch. along  $u = u_1$ ,  $V^+$  to  $M$  along  $u = u_0$ ); the singularity at  $r = 0$  (double line) becomes the origin of coordinates in  $M$  (dashed line) when the mass has dropped to zero (at  $v = v_0$ ).

$S'' = (u = u_0, v = v_0)$ . In the union of the regions to the past of  $\mathcal{S}'$  and to the past of  $\mathcal{S}''$ , retain the Schwarzschild metric.

There will necessarily be distributional curvature along  $\Sigma$ , and the choices of  $N(v)$ ,  $M(u)$ , this distributional curvature and the history of  $\Sigma$  are tied together by the Einstein equations. However, for our purposes, we don't need the details of this as long as there is a sphere ( $u = u_0, v = v_0$ ) on  $\Sigma$  as a future boundary at which  $N(v_0) = 0 = M(u_0)$ . Assume so, and then match to flat space across  $v = v_0$  and across  $u = u_0$ . The Penrose diagram for this metric is given in figure 1.

Note in particular that  $\Sigma$  must stop before the singularity is reached. This is because at the singularity  $r$  vanishes, while in the outgoing Vaidya metric the singularity at  $r = 0$  is in the past, corresponding to the past singularity of Schwarzschild, not the future one.

Our interest is in the future singularity of the Schwarzschild metric at  $r = 0$  (the past one is in a part of Schwarzschild not relevant in collapse). Points of the future one with  $v \leq v_1$  are just as in Schwarzschild, while points with  $v_1 \leq v < v_0$  are as in Vaidya. Our main interest is in points with  $v = v_0$ , and the approach to them along different radial null directions. When approached from the past like this, the last part of the approach is solely through the part of the metric which is ingoing Vaidya; approach from the future is through flat space; therefore we don't need to calculate with the part of the metric which is outgoing Vaidya, and so the details of  $\Sigma$  don't affect our calculation.

The ingoing Vaidya metric is

$$ds^2 = \left(1 - \frac{2N(v)}{r}\right) dv^2 - 2dvdr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

and, although we shan't need it, the outgoing Vaidya metric is

$$ds^2 = \left(1 - \frac{2M(u)}{r}\right) du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (2)$$

We choose the following Newman-Penrose (NP) tetrad for the ingoing Vaidya metric:

$$D = \partial_v + \frac{1}{2} \left(1 - \frac{2N(v)}{r}\right) \partial_r, \quad \Delta = -\partial_r, \quad (3)$$

$$\delta = \frac{1}{r\sqrt{2}} \left( \partial_\theta + \frac{i}{\sin\theta} \partial_\phi \right),$$

so that  $\ell^a$  is outgoing into the future and  $n^a$  is ingoing. Now calculate, using the NP formalism

$$D\ell^a = 2\epsilon\ell^a, \quad Dn^a = -2\epsilon n^a, \quad (4)$$

and

$$\Delta\ell^a = 0, \quad \Delta n^a = 0, \quad (5)$$

where

$$\epsilon = \frac{N}{2r^2}. \quad (6)$$

Note also that

$$\rho = -\frac{Dr}{r} = -\frac{1}{2r} \left(1 - \frac{2N(v)}{r}\right), \quad (7)$$

so that the surface  $r = 2N(v)$  consists of marginally outer-trapped surfaces of constant  $r$ , interpolating between the horizon of the initial Schwarzschild solution at  $r = 2m$  on  $v = v_S$  and the origin of flat space at  $v = v_0$ . This surface is a dynamical horizon in the language of [1], but is diminishing in size as it has a flux of negative energy across it.

The only nonzero curvature components are

$$\Phi_{00} = \frac{N'}{r^2}, \quad \Psi_2 = -\frac{N}{r^3}. \quad (8)$$

so that

$$\Phi_{ab} = \Phi_{00}n_a n_b, \quad \Psi_{ABCD} = 6\Psi_2 o_{(A} o_B l_C l_{D)}, \quad (9)$$

in terms of the spinor dyad underlying the chosen NP tetrad.

Hiscock [3], [4] proposes imposing the condition  $N'(v_0) = 0$ , as this is necessary and sufficient to give a finite rate of particle production at  $v = v_0$ . An alternative would be to take

$$N(v) \sim (v_0 - v)^{1/3}, \quad (10)$$

which is the dependence obtained if one assumes that the Hawking expression for the power radiated by an evaporating black hole holds up to the end. In this case, the Ricci tensor is singular at  $v_0$ . We'll return to this point below, but note here that, away from  $r = 0$ , Hiscock's condition has the effect of making the curvature continuous.

### 3 Conformal geodesics and the method of analysis

A null conformal geodesic (see e.g. [11]) is a curve  $\gamma$  with tangent vector  $v^a$  and a covector  $b_a$  defined along  $\gamma$  satisfying the coupled system

$$\nabla_v v^a + 2(b_c v^c) v^a = 0 \quad (11)$$

$$\nabla_v b_a - (b_c v^c) b_a + \frac{1}{2}(b_c b^c) v_a = P_{ab} v^b \quad (12)$$

where  $P_{ab} = \Phi_{ab} - \Lambda g_{ab} = -\frac{1}{2}R_{ab} + \frac{1}{12}Rg_{ab}$  is the Rho or Schouten tensor (here we use the conventions of [9] since we use the spin-coefficient formalism; other authors may have permutations of the signs here). By (11), a null conformal geodesic is a null metric geodesic with a particular scaling, dictated by a chosen solution of (12).

Define a propagation law along conformal geodesics by

$$\nabla_v e^a + (b_c v^c) e^a + (b_c e^c) v^a - (e_c v^c) b^a = 0, \quad (13)$$

which as we shall see has a good transformation under conformal rescaling, then in particular  $v^a$  is carried along by this rule. For spinors the propagation (13) reduces to

$$\nabla_v \alpha^A = -v^{AA'} b_{A'C} \alpha^C \quad (14)$$

Under conformal rescaling

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \quad (15)$$

we claim that solutions of (11)-(13) transform according to

$$\tilde{v}^a = v^a, \tilde{b}_a = b_a - \Upsilon_a, \tilde{e}^a = e^a,$$

where, as usual,  $\Upsilon_a = \Omega^{-1} \nabla_a \Omega$ , and then (14) is invariant with  $\tilde{\alpha}^A = \alpha^A$ . Thus (14) provides a conformally-invariant propagation of, say, a suitably normalised spinor dyad along  $\gamma$ , from which we construct a conformally-invariant test for boundedness of Weyl spinor components.

For the normalisation, note that, given two solutions  $\alpha_1^A$  and  $\alpha_2^A$  of (14)

$$\nabla_v (\epsilon_{AB} \alpha_1^A \alpha_2^B) = -(b_c v^c) \epsilon_{AB} \alpha_1^A \alpha_2^B$$

so that, if  $\Omega$  is a conformal factor with

$$v^a \tilde{b}_a := v^a (b_a - \Upsilon_a) = 0 \quad (16)$$

then

$$\tilde{\epsilon}_{AB} \alpha_1^A \alpha_2^B := \Omega \epsilon_{AB} \alpha_1^A \alpha_2^B$$

is constant along the conformal geodesic. This rescaling of  $\epsilon_{AB}$  corresponds to the rescaling (15) of the metric and then (11) becomes

$$\tilde{\nabla}_v v^a = 0,$$

where  $\tilde{\nabla}$  is the metric covariant derivative for  $\tilde{g}$ , so that  $v^a$  is tangent to affinely-parametrised null metric geodesics for  $\tilde{g}$ . It is then easy to see that  $u^a = \Omega^2 v^a$  is affinely-parametrised for  $g$ . If  $s$  and  $\tilde{s}$  are affine parameters for  $u$  and  $v$  respectively then

$$1 = \nabla_v \tilde{s} = \Omega^{-2} \nabla_u \tilde{s} = \Omega^{-2} \frac{d\tilde{s}}{ds}. \quad (17)$$

Note that  $\tilde{s}$  is a projective parameter along  $\gamma$  in the usual terminology of conformal geodesics [11], so that different choices of  $v^a$  and  $b_a$  subject to (11) and (12) lead to Möbius transformations in  $\tilde{s}$ . Equation (16) can be rewritten as

$$\Omega^{-1}\nabla_v\Omega = b_cv^c$$

and then the contraction of (12) with  $v^a$  gives

$$\nabla_v\nabla_v\Omega^{-1} = \Omega^{-1}(P_{ab}v^av^b). \quad (18)$$

The method is now as follows: an affinely-parametrised null geodesic  $\gamma$  of the space-time metric, with tangent  $u^a$ , gives rise to a null conformal geodesic by solving (12) along  $\gamma$  for  $b_a$ , with  $v^a$  replaced by  $u^a$ , and then adjusting the scaling by solving (11) for  $v^a$ ; one can use (17) to see if the null geodesic is incomplete for  $\tilde{s}$ ; if it is, one can use a spinor dyad solving (14) to test the curvature components for boundedness on  $\gamma$ . If  $\gamma$  is incomplete for  $\tilde{s}$  but with bounded curvature for  $\tilde{g}$ , then we expect to be able to extend the unphysical metric in a neighbourhood of  $\gamma$  including a final segment [11], [6]. This method therefore tests for extendibility of the conformal metric.

We shall apply this method to ingoing and outgoing radial null geodesics in the ingoing Vaidya metric which encounter the singularity at  $r = 0$ , to test whether the conformal metric can ever be extended through this singularity.

## 4 The analysis

### 4.1 In-going null geodesics

First we consider ingoing null geodesics, that is null geodesics parallel to  $n^a$ . This geodesic vector field is affinely-parametrised for the physical metric (by (5)) so we take  $u^a = n^a$ . Note that the coordinate  $r$  is an affine parameter, so that the singularity is at a finite distance in the physical affine parameter, and the coordinate  $v$  is constant along such a  $\gamma$ . We distinguish  $v = v_2 < v_0$ , an ingoing null geodesic meeting the singularity in its interior, from  $v = v_0$ , an ingoing null geodesic meeting the ‘end’ of the singularity. If  $v_2 < v_1$ , then we are testing the Schwarzschild metric for extendibility, while  $v_1 < v_2 \leq v_0$  relates to the Vaidya metric.

Expand  $b_a$  in the NP tetrad as

$$b_a = X\ell_a + Yn_a - \bar{Z}m_a - Z\bar{m}_a$$

and expand (12) using (5) and (8) to obtain the system

$$\begin{aligned} \Delta X &= X^2 \\ \Delta Y &= Z\bar{Z} \\ \Delta Z &= XZ. \end{aligned}$$

We won’t need  $Y$ . Note that, from (16),

$$\Omega^{-1}\Delta\Omega = b_cu^c = X$$

from which we obtain  $\Omega$ , given  $X$ .

This system of equations is homogeneous (since, by (9),  $P_{ab}n^b = 0$ ) so that the solutions are as in vacuum. For  $X$  and  $Z$ , use (3) to find two cases:

1.  $X = 0$ ,  $Z = Z_0$  for constant  $Z_0$ , so  $\Omega = 1$  w.l.o.g.; or
2.  $X = (r + r_0)^{-1}$ ,  $Z = Z_0(r + r_0)^{-1}$  for constants  $r_0$  and  $Z_0$ , so  $\Omega = (r + r_0)^{-1}$ .

Now for the spinor propagation (14), expand

$$\alpha^A = \zeta o^A + \eta \iota^A$$

to find

$$\begin{aligned}\Delta\zeta &= 0 \\ \Delta\eta &= -X\eta - Z\zeta.\end{aligned}$$

A basis of solutions in the two cases is given by

1.  $O^A = o^A + (\eta_0 + rZ_0)\iota^A$ ,  $I^A = \iota^A$
2.  $O^A = o^A + (\eta_0(r + r_0) - Z_0)\iota^A$ ,  $I^A = (r + r_0)\iota^A$ .

We wish to test the components of the Weyl spinor in these bases. From (9) in case 1 we find

$$\psi_2 = \Psi_2 = -\frac{N}{r^3}$$

which clearly diverges on the approach to the singularity at  $r = 0$ . In case 2, we have

$$\psi_2 = (r + r_0)^2 \Psi_2$$

and, for any choice of  $r_0$ , this still diverges if  $v = v_2 < v_0$ , (which, by the remark above, includes the Schwarzschild singularity) but it will be zero on  $v = v_0$ . To go further with the case  $v = v_0$ , we calculate the other part of the tractor curvature, which is the Cotton tensor  $\nabla_{[a}P_{b]c}$  and which will be conformally invariant at points where the Weyl tensor vanishes. For this Vaidya metric, it is given by

$$\nabla_{[a}P_{b]c} = \nabla_{[a}\Phi_{b]c} = N'\nabla_{[a}\left(\frac{1}{r^2}n_{b]}n_c\right),$$

with the aid of (8) and (9), which is

$$\frac{N'}{r^3} (3\ell_{[a}n_{b]}n_c + n_{[a}g_{b]c})$$

The component of this along  $I^A\bar{I}^{A'}O^BO^C\bar{O}^{B'}\bar{O}^{C'}$  is  $N'(v_0)/r^3$  in case 1 and  $N'(v_0)\{(r + r_0)^2/r^3\}$  in case 2, which diverges in both cases unless  $N'(v_0) = 0$ . The vanishing of this, which is necessary for finiteness of the tractor curvature, is Hiscock's condition [3], [4] for finite rates of particle creation at the end of the black-hole evaporation, so that there is a physical motivation for imposing it. If we do, then the whole of the tractor curvature is zero on this null geodesic (If we used (10) instead, then the tractor curvature is singular and no extension is possible.)

To test for the existence of a conformal extension, we need to consider higher derivatives of the tractor curvature. We next calculate the Bach tensor:

$$B_{ab} = 2\nabla_{A'}^C\nabla_{B'}^D\Psi_{ABCD} + 2\Psi_{ABCD}\Phi_{A'B'}{}^{CD}.$$



Recall that, under (15),  $\tilde{B}_{ab} = \Omega^{-2}B_{ab}$ . When  $v = v_0$ , if  $N = N' = 0$ , the only nonzero term in  $B_{ab}$  will be

$$\begin{aligned} & -\frac{12}{r^3}o_{(A}o_B\iota_C\iota_{D)}\nabla_{A'}^C\nabla_{B'}^DN(v) \\ & = -\frac{2}{r^3}n_an_bN'', \end{aligned}$$

using (3). Now components of  $\tilde{B}_{ab}$  in frames of either case diverge unless  $N'' = 0$ , so there is no conformal extension possible unless  $N'' = 0$ . Inductively, if  $N^{(k)}(v_0) = 0$  for  $0 \leq k \leq n-1$  then the derivatives

$$\nabla_{a_1} \cdots \nabla_{a_k} C_{bcd}{}^e$$

will be zero for  $0 \leq k \leq n-1$ , and the case  $k = n$  will be a conformally-invariant tensor, whose only nonzero term will be proportional to  $N^{(n)}r^{-3}$ ; so the  $(n-1)$ -th derivative of conformal (or tractor) curvature will be bounded if and only if  $N^{(n)}(v_0) = 0$ , and then a local  $C^{n-2}$ -extension of the conformal structure will exist, by [6].

## 4.2 Outgoing null geodesics

Next we consider outgoing null geodesics, in the direction of  $\ell^a$ . This vector field is not affinely-parametrised (see (4)), so we introduce  $u^a = \Theta^2\ell^a$  which is. By (4), the condition on  $\Theta$  is

$$0 = \nabla_u u^a = \Theta^2 D(\Theta^2 \ell^a),$$

so that

$$D\Theta = -\epsilon\Theta. \tag{19}$$

Again, we expand  $b_a$  in the NP tetrad, to find for (12) the system

$$\begin{aligned} DX + 2\epsilon X - Z\bar{Z} &= 0 \\ DY - 2\epsilon Y - Y^2 &= \frac{N'}{r^2} \end{aligned} \tag{20}$$

$$DZ - YZ = 0. \tag{21}$$

and for the spinor propagation

$$D\zeta + \epsilon\zeta = -Y\zeta - \bar{Z}\eta \tag{22}$$

$$D\eta - \epsilon\eta = 0. \tag{23}$$

For the conformal factor

$$\Omega^{-1}D\Omega = b_c\ell^c = Y, \tag{24}$$

Use (24) to solve (21) and find  $Z = \Omega Z_0$ , then (20) becomes a linear equation for  $\Omega^{-1}$  :

$$D^2\Omega^{-1} + 2\Theta^{-1}D\Theta D\Omega^{-1} + \Omega^{-1}\frac{N'}{r^2} = 0 \tag{25}$$

which is equivalent to (18). Using (7), one solution is found to be  $\Omega = r^{-1}$ , corresponding to  $Y = \rho$ . The general solution can therefore be written as

$$\Omega^{-1} = rF \tag{26}$$

whereupon (25) can be integrated to give

$$DF = Ar^{-2}\Theta^{-2} \quad (27)$$

for constant  $A$ .

From (19) we may solve (23) by

$$\eta = \eta_0\Theta^{-1}.$$

Now use (19) and (24) to convert (22) to

$$D(\Theta^{-1}\Omega\zeta) = -\bar{Z}_0\eta_0\Omega^2\Theta^{-2}.$$

A normalised spinor dyad is

$$\begin{aligned} O^A &= \Theta\Omega^{-1}o^A \\ I^A &= W o^A + \Theta^{-1}t^A \end{aligned}$$

where

$$D(\Theta^{-1}\Omega W) = -\bar{Z}_0\Omega^2\Theta^{-2}.$$

In this dyad, the nonzero Weyl curvature components are

$$\psi_2 = \Omega^{-2}\Psi_2, \quad (28)$$

$$\psi_3 = 3\Theta^{-1}W\Omega^{-1}\Psi_2, \quad (29)$$

$$\psi_4 = 6W^2\Theta^{-2}\Psi_2. \quad (30)$$

We need the behaviour of these along  $\gamma$ , where

$$\frac{dr}{dv} = \frac{1}{2} - \frac{N(v)}{r},$$

which we need to solve for  $r(v)$ . We distinguish two cases:

- $v \rightarrow v_2 < v_0$ : this includes the case of approach to the Schwarzschild singularity.

Then

$$r = (2N(v_2))^{1/2}(v_2 - v)^{1/2}(1 + O(v_2 - v)),$$

so

$$\Theta = (v_2 - v)^{1/4}(1 + O(v_2 - v)).$$

From (26)-(27),  $\Omega^{-1}$  is therefore asymptotically  $O(1)$  or  $O((v_2 - v)^{1/2})$  as  $v \rightarrow v_2$ . From (28) and (8), we can't make  $\psi_2$  bounded.

From the definition of  $\Theta$ , the physical affine parameter  $s$  is obtained by solving

$$\frac{ds}{dv} = Ds = \Theta^{-2},$$

so that

$$\int ds = \int \Theta^{-2} dv. \quad (31)$$

Now the distance to the singularity in the physical affine parameter is finite.

- $v \rightarrow v_0$ : now we need to make an assumption about the behaviour of  $N(v)$  subject to  $N(v_0) = 0$ ; we shall suppose that

$$N(v) = n(v_0 - v)^k$$

with  $k > 1$ , so that Hiscock's condition  $N'(v_0) = 0$  holds, then along  $\gamma$  we calculate

$$r = 2n(v_0 - v)^k (1 - 4nk(v_0 - v)^{k-1} + O((v_0 - v)^{2(k-1)})).$$

Solve (19), using (6), to find

$$\Theta = (v_0 - v)^k \exp \left( -\frac{1}{8n(k-1)(v_0 - v)^{k-1}} \right) (1 + O((v_0 - v)^{k-1})). \quad (32)$$

From (31) and (32), the integral for  $s$  diverges as  $v \rightarrow v_0$ , so that this singularity is infinitely far away along  $\gamma$ , as measured in the physical affine parameter. Note that this is different from the previous case of radial null geodesics ingoing towards this same singularity.

From (26), (28) and (8):

$$\psi_2 = -\frac{F^2 N}{r} = -\frac{F^2}{2} (1 + O((v_0 - v)^{k-1})),$$

while, from (27)

$$\frac{dF}{dv} = \frac{A'}{(v_0 - v)^{4k}} \exp \left( \frac{1}{4n(k-1)(v_0 - v)^{k-1}} \right) (1 + O((v_0 - v)^{k-1})) \quad (33)$$

with  $A' = A(4n^2)^{-1}$ .

We distinguish two cases. If  $A = 0$  then  $F$  is constant and  $\psi_2$  is finite. By choosing  $Z_0 = 0 = W$ , we ensure that the whole Weyl spinor is bounded (from (29), (30)). From (17) however, choosing  $F = 1$  we find that

$$\frac{d\tilde{s}}{dv} = r^{-2} \Theta^{-2},$$

and the right-hand-side of this is the same as in (33), up to a nonzero factor. This equation has no solution which is bounded as  $v \rightarrow v_0$ , so that the singularity is infinitely far off in the rescaled, unphysical metric - no conformal extension is possible.

Now if  $A \neq 0$  then we have to solve (33) for  $F$  and again this has no solution which is bounded as  $v \rightarrow v_0$ , so that  $\psi_2$  cannot be made finite - in this case the distance to the singularity in the unphysical affine parameter is finite since

$$\frac{d\tilde{s}}{dv} = \Omega^2 \Theta^{-2} = r^{-2} F^{-2} \Theta^{-2} = A^{-1} F^{-2} \frac{dF}{dv},$$

so that

$$\tilde{s} = c_1 + \frac{c_2}{F},$$

which is bounded, but the Weyl curvature is singular so that no conformal extension is possible this way either. With Hiscock's condition, no local conformal extension is possible around the outgoing null geodesic to  $v = v_1$ .

It is straightforward to repeat the last calculation with (10) instead. Now the physical affine parameter distance to the singularity is finite, but no choices lead to finite Weyl spinor, so that no local conformal extension is possible around this null geodesic.

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